6.2 Elements of excitonic states and absorption

The theory of excitons is based on a many-body formalism and an outline of this will be given in a later section. In the present section we discuss the basic concept first and then develop the expressions for absorption coefficients by using hydrogenic states obtained from a two-particle Schrödinger equation.
6.2.1 A crude hydrogenic model

We first develop some useful excitonic parameters by using a simple intuitive picture. Consider a single electron and a single hole that are created in the semiconductor by the process of photon absorption. If the particles are describable by their respective effective masses, then the hole is heavier than the electron. Assuming first that the hole is stationary, the Coulomb attraction between the electron and the hole will cause the negatively charged electron to revolve around the hole just as in a hydrogen atom. The hole in this case represents the positively charged nucleus in the hydrogen atom. Actually the picture is more like the positronium atom composed of an electron and a positron. We may apply Bohr's theory of the hydrogen atom to calculate the energy levels, by introducing the following modifications. First, the holes are not stationary, but have a relative motion. The effective reduced mass describing the relative motion of the nucleus and electron (see, for example, Merzbacher 1970; p. 347) should be included in the expression for the energy levels. Secondly the Coulomb attraction should be modified by the permittivity of the material. Thus in analogy with the hydrogen atom problem, there exists a whole series of eigenstates labelled 1s, 2s, 2p, ..., and the energies are given by

$$E_{n0} = E_g - E_{Bn}. \tag{6.1}$$

The binding energy of the $n$th level is given by this intuitive picture as

$$E_{Bn} = \frac{\mu e^4}{32\pi^2 \epsilon^2 \hbar^2 n^2} = \frac{\hbar^2}{2\mu a_B^2 n^2}. \tag{6.2}$$
Here the energies are measured from the top of the valence band and when the energies \( E_{n0} \) exceed the gap energy, one encounters the ionization continuum as in the hydrogen atom problem. Notice the appearance of the reduced mass \( \mu \) \((\mu^{-1} = m_e^{-1} + m_h^{-1})\) and the material permittivity \( \varepsilon \) in eqn (6.2). The symbol \( a_B \) denotes the Bohr radius and is expressed as

\[
a_B = \frac{4\pi\varepsilon h^2}{\mu e^4}.
\]

(6.3)

**Example 6.1** In the hydrogen atom \( E_{B1} = 13.6 \text{ eV} \), and \( a_B = 0.53 \text{ Å} \) for the 1s \((n = 1)\) level. Take, now, the values for GaAs as an example: \( m_e = 0.067m_0 \) and \( m_h = 0.5m_0 \), \( \varepsilon = 13\varepsilon_0 \). Thus \( E_B = 4.6 \text{ meV} \) and \( a_B = 120 \text{ Å} \). Since the lattice constant is 5.6 Å, the exciton Bohr radius is spread over many lattice sites. The use of the effective mass in describing the Wannier exciton is therefore justified.
6.2.2 Two-particle wave equation

The band-to-band absorption in the previous chapter was formulated with the help of the single-electron picture. In the present problem, two particles are involved, namely an electron and a hole. It is therefore necessary to use a two-particle Schrodinger equation that involves a two-particle wavefunction $\phi(r_e, r_h)$, where $r_e$ and $r_h$ are the coordinates of the electron and hole, respectively.

In the ground state of the crystal, all the states in the valence band are full. An excited state is formed by removing an electron from state $|vk\rangle$ and placing it in the state $|ck'\rangle$ in the conduction band. We use

$$\varphi_{ck_e}(r_e) = \varphi_{ck'}(r_e) = U_e \exp(i k_e \cdot r_e)$$

(6.4a)

for the wavefunction in the conduction band, and

$$\varphi_{vk_h}(r_h) = \varphi_{vk_h}^*(r_h) = U_v \exp(i k_h \cdot r_h),$$

(6.4b)

as the wavefunction for the hole in the valence band. The total wavevector of the excited state is

$$K = k' - k = k_e + k_h.$$  

(6.5)

Let us first investigate the nature of the two-particle wavefunction, ignoring the Coulomb interaction between the pairs. The equation may be obtained by simply adding the effective mass equations for a free electron and a free hole. For the former this is

$$H_e \varphi_e(r_e) = E_e \varphi_e(r_e),$$

(6.6a)

with

$$H_e = E_e - \frac{\hbar^2}{2m_e} \nabla_e^2.$$  

(6.6b)
For a free hole, the effective mass equation is
\[ H_v \varphi_h(r_h) = -\frac{\hbar^2}{2m_h} \nabla^2_h = -E_v \varphi_h(r_h). \] (6.7)

The two particle effective-mass equation is obtained by writing
\[ (H_c + H_v) \varphi(r_e, r_h) = E \varphi(r_e, r_h). \] (6.8)

It is straightforward to show that in the case of no interaction the two-particle wavefunction is \( \varphi(r_e, r_h) = \varphi_e(r_e) \varphi_h(r_h) \) and one obtains eqns (6.6) and (6.7) by separation of variables.

We now include the electron–hole Coulomb interaction and obtain the two-particle effective-mass equation as follows:
\[ \left( H_c + H_v - \frac{e^2}{4\pi \varepsilon |r_e - r_h|} \right) \phi(r_e, r_h) = E \phi(r_e, r_h). \] (6.9)

Equation (6.9) is exact for distinguishable particles like the proton and electron (see Merzbacher 1970; p. 347). When indistinguishable particles are involved, there is also an exchange interaction.

The previous method of using a separable product \( \varphi_e \varphi_h \) is now no longer valid. Still, separation may be accomplished if the centre-of-mass \( (R) \) and relative \( (r) \) coordinates are defined as
\[ R = \frac{m_e r_e + m_h r_h}{M}, \quad M = m_e + m_h, \] (6.10)
\[ r = r_e - r_h. \]
Corresponding to eqn (6.10) are the centre-of-mass \( (K) \) and relative \( (k) \) wave-vector given as follows:

\[
K = k_e + k_h, \tag{6.11a}
\]

\[
k = \frac{m_e k_e - m_h k_h}{M}, \tag{6.11b}
\]

so that

\[
k_e \cdot r_e + k_h \cdot r_h = k \cdot R + k \cdot r. \tag{6.11c}
\]

Now

\[
\frac{\partial}{\partial x_e} = \frac{\partial}{\partial x} \frac{\partial x}{\partial x_e} + \frac{\partial}{\partial X} \frac{\partial X}{\partial x_e} = \frac{\partial}{\partial x} + \frac{m_e}{M} \frac{\partial}{\partial X}.
\]

Thus

\[
\frac{\partial^2}{\partial x_e^2} = \frac{\partial^2}{\partial x^2} + \left(\frac{m_e}{M}\right)^2 \frac{\partial^2}{\partial X^2} + \frac{2m_e}{M} \frac{\partial^2}{\partial x \partial X}.
\]

Similarly

\[
\frac{\partial^2}{\partial x_h^2} = \frac{\partial^2}{\partial x^2} + \left(\frac{m_h}{M}\right)^2 \frac{\partial^2}{\partial X^2} - \frac{2m_h}{M} \frac{\partial^2}{\partial x \partial X}.
\]

Using these and similar relations for the \( y \)- and \( z \)-coordinates in eqn (6.9), and making use of eqns (6.6) and (6.7) and the definitions of the total mass \( M \) and the reduced mass \( \mu \), one obtains

\[
\left( E - \frac{\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{e^2}{4\pi \epsilon r} \right) \phi(R, r) = E \phi(R, r). \tag{6.12}
\]
Now that the centre-of-mass and relative coordinates are decoupled, we employ a product wavefunction of the form
\[ \phi(R, r) = g(R)F(r). \]  
(6.13)

Using eqn (6.13) in eqn (6.12), one obtains two equations, viz.
\[ -\frac{\hbar^2}{2M} \nabla_R^2 g(R) = E_K g(R), \]  
(6.14a)

and
\[ \left( E_g - \frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{e^2}{4\pi\epsilon_0 r} \right) F(r) = E_0 F(r), \]  
(6.14b)
\[ E = E_0 + E_K. \]  
(6.14c)

Equation (6.14a) is the Schrödinger equation for a free particle. It shows that the centre of mass of the pair remains unaffected by the mutual attraction. The solutions are plane waves
\[ g(R) = \frac{1}{\sqrt{1/2}} e^{iK \cdot R}. \]  
(6.15)

The corresponding eigenvalues \( E_K \) are given by
\[ E_K = \frac{\hbar^2 K^2}{2M}. \]  
(6.16)

\( E_K \) thus represents the kinetic energy due to the motion of the centre of mass.
6.2.3 Hydrogenic states

We rewrite eqn (6.14b) as

\[
\left( -\frac{\hbar^2}{2\mu} \nabla r^2 - \frac{e^2}{4\pi \epsilon r} - E_n \right) F_n(r) = 0, \tag{6.17}
\]

where the \( n \)'s label different eigenstates. We may use the hydrogenic wavefunctions to describe the functions \( F_n(r) \), so that

\[
F_n(r) = R_{n,l}(r) Y_l^m(\theta, \phi) \tag{6.18}
\]

in spherical polar coordinates. The eigenvalues \( E_n \) may be either positive or negative. The negative values of \( E_n \) (for states in the forbidden gap) correspond to bound states and the positive values (above the band gap) give rise to continuum states. We give below the eigenfunctions. Details of the calculation may be found in textbooks of quantum mechanics (see, for example, Landau and Lifshitz 1975; p. 117). The radial part \( R_{n,l}(r) \) may be written for both discrete and unbound states as

\[
R_{n,l}(r) = N_n r^l e^{-\rho/2} F(l + 1 - \eta; 2l + 2; \rho), \tag{6.19}
\]

where \( N_n \) is the normalization constant, \( F \) is the confluent hypergeometric function, and \( \rho \) is related to \( r \). \( \eta \) is an energy parameter defined as

\[
\eta^2 = -\frac{R_x}{E}; \quad R_x = \frac{\hbar^2}{2\mu a_B^2}. \tag{6.20}
\]
Discrete states \((E < 0)\) In this case \(\eta\) is the principal quantum number \(n\).

\[
E_n = -\frac{R_x}{n^2},
\]  
(6.21a)

\[
\rho = \frac{2r}{na_B},
\]  
(6.21b)

\[
N = \frac{2/na_B}{(2l + 1)!} \frac{(n + l)!}{(n - l - 1)! 2n}
\]  
(6.21c)

and

\[
|F_n(0)|^2 = (\pi a_B^3 n^3)^{-1}.
\]  
(6.21d)

It follows therefore that only s-states are non-zero at \(r = 0\).
Continuum states \((E_n > 0)\) When the energy \(E\) becomes positive, \(\eta\) must be an imaginary quantity. We write

\[
\eta = i\gamma = \frac{i}{k a_B}, \tag{6.22a}
\]

\[
E(k) = -\frac{R_x}{(i\gamma)^2} = R_x (k a_B)^2 = \frac{\hbar^2 k^2}{2\mu}, \tag{6.22b}
\]

\[
\rho = -2ikr, \tag{6.22c}
\]

\[
N = (i^l/(2^l + 1)!)[\Gamma(l + 1 - i\gamma)]e^{\pi\gamma/2} \tag{6.22d}
\]

and

\[
|F(0)|^2 = z \frac{\exp(z)}{\sinh z}, \quad z = \pi\gamma. \tag{6.22e}
\]

From eqn (6.22b)

\[
E(k) = \frac{R_x}{\gamma^2} = \hbar\omega - E_g, \tag{6.22f}
\]

\[
\gamma = \left[\frac{R_x}{\hbar\omega - E_g}\right]^{1/2}. \tag{6.22g}
\]

It should be noted that \(k\) (or \(\gamma\)) is nothing but a quantum number and has nothing to do with momentum. Although the energy is \(E(k) = \hbar^2 k^2/2\mu\), it is composed of both potential and kinetic energy contributions of the relative motion.
electron-hole matter: two band model

vacuum full valence band

\[ |\Phi_{HL}^{0}\rangle = \left(\prod_{v,\vec{k}} c^\dagger_{v,\vec{k}}\right)|\Phi_{0}\rangle \]

electrons \quad a^\dagger_{\vec{k}} = c^\dagger_{c,\vec{k}} \quad \text{holes:} \quad b^\dagger_{\vec{k}} = c_{v,-\vec{k}}

excited semiconductor

\[ \left(\prod_{\vec{k}} f^{e}_{e,\vec{k}} a^\dagger_{\vec{k}}\right) \left(\prod_{\vec{k}} f^{h}_{h,\vec{k}} b^\dagger_{\lambda,\vec{k}}\right)|\Phi_{HL}^{0}\rangle \]

Hamilton operator

\[ H = H_0 + H_{Coul} + H_{rad} \]

\[ H_0 \quad \text{single particle} \]
\[ H_{Coul} \quad \text{Coulomb interaction between carriers} \]
\[ H_{rad} \quad \text{radiative coupling to optical field} \]
**single e-h pair: exciton**

**K-space picture**

e-h pair at wave vector $\vec{K}$

superposition of many pairs with identical $\vec{K}$

exciton dispersion

$$E_\alpha(\vec{K}) = E_\alpha + \frac{\hbar^2 K^2}{2M^*} + E_g$$

$$E_\alpha = -\frac{E_X}{n^2}$$

“hydrogen” series

$$E_X = \frac{\mu^*}{\epsilon^2} E_B$$

size: exciton Bohr radius

$$a_X = \frac{\epsilon}{\mu^*} a_B$$

$2 - 10nm$
\[ B_{\nu, K}^\dagger = \sum_{k,k'} \delta[K - (k - k')] \psi_{\nu} \left( \frac{k + k'}{2} \right) \alpha^{\dagger}_{k} \beta^{\dagger}_{-k'} \]

\[ = \sum_{k} \psi_{\nu} (k - K/2) \alpha^{\dagger}_{k} \beta^{\dagger}_{K-k} \]

\[ \psi_0(k) = 8 \sqrt{\pi a_0^3} \frac{1}{\left[1 + (ka_0)^2\right]^2} \] in 3D

The commutator for the exciton operators is

\[ [B_{0,0}, B_{0,0}^\dagger] = \sum_{k,k'} \psi_0(k) \psi_0^*(k') \left[ \beta_{-k} a_k, \alpha^\dagger_k, \beta^\dagger_{-k'} \right] \]

\[ = \sum_k |\psi_0(k)|^2 (1 - \alpha^\dagger_k a_k - \beta^\dagger_{-k} \beta_{-k}) \]
6.2.5 Calculation of the absorption coefficient

Squared matrix elements We shall first give a simplified picture (Datta 1989) for the calculation of the absorption coefficient using the two-particle wavefunction derived from an intuitive two-particle effective-mass equation. The wavefunction may be written by ignoring the cell-periodic part of the Bloch functions as

$$\phi(R, r) = \frac{e^{iK \cdot R} e^{i k \cdot r}}{\sqrt{V} \sqrt{V}}. \quad (6.24)$$

In optical absorption, electron-hole pairs with $k_e = k_h$ are created, so that the centre-of-mass momentum of the exciton is zero ($K = 0$). The relative momentum is however non-zero.

In band-to-band absorption processes, the absorption coefficient is calculated by summing over a matrix element as follows:

$$\kappa = \sum_k |M_{cv}(k)|^2 \delta(E - \hbar \omega).$$

In the present situation, instead of summing over $k$, we need a sum over the index $n$. To calculate $M_{cv}(n)$ in terms of the matrix $M_{cv}(k)$, we note that the new eigenstates $F_n(r)$ given by eqn (6.18) may be written as a linear combination of old plane-wave functions as

$$F_n(r) = \sum_k \phi_{n,k} e^{i k \cdot r} \frac{1}{\sqrt{V}}. \quad (6.25)$$
In order that $F_n(r)$ is normalizable, one should have $\sum |\phi_{nk}|^2 = 1$. The new matrix elements $M_{cv}(n)$ are linear combinations of the old matrix elements

$$M_{cv}(n) = \sum_k \phi_{nk} M_{cv}(k) = M_{cv}(0) \sum_k \phi_{nk}.$$  \hspace{1cm} (6.26)

We have assumed that $M_{cv}(k) \simeq M_{cv}(0)$. We may now write, from eqn (6.25),

$$\sum_k \phi_{nk} = \sqrt{V} F_n(r = 0).$$  \hspace{1cm} (6.27)

Therefore

$$M_{cv}(n) = M_{cv}(0) \sqrt{V} F_n(0).$$  \hspace{1cm} (6.28)

Substituting this in the expression for $\alpha$ one may now obtain

$$\alpha(h\omega) = \frac{2\pi \langle |p_{cv}|^2 \rangle}{\eta \epsilon_0 \omega_c} \left( \frac{e}{m_0} \right)^2 \sum_n |F_n(0)|^2 \delta \left( h\omega - E_g + \frac{R_x}{n^2} \right)$$  \hspace{1cm} (6.29)

for discrete states.

**Absorption coefficients** Equation (6.29) indicates that the envelope function $F(r = 0)$ is involved in the calculation of $\alpha(h\omega)$. Since s-states have non-zero values at $r = 0$, only these states are important in the absorption processes. It follows from eqn (6.21d) that the absorption coefficients decrease as $n^{-3}$ with an increase in the value of the quantum number $n$. 
For higher values of $n$, the excitonic lines overlap and the result is a quasi-continuous absorption. We may calculate $\alpha$ as $\hbar \omega \rightarrow E_g$ by taking $n$ as a continuous variable and replacing the sum by an integral over $n$. Using $x = \hbar \omega - E_g + R_x/n^2$, and $dx = -2R_x \,dn/n^3$, and noting that $|F_n(0)|^2 = (\pi a_B^3 n^3)^{-1}$, we may write

$$\lim_{\hbar \omega \rightarrow E_g} \alpha(\hbar \omega) = \lim_{\hbar \omega \rightarrow E_g} \frac{2(e/m_0)^2}{\eta \epsilon_0 \omega a_c (a_B)^3} \left< |p_{cvl}|^2 \right> \int_{\hbar \omega \rightarrow E_g} dx \frac{\delta(x)}{R_x} \frac{2 \pi \sqrt{R_x}}{\sqrt{\hbar \omega - E_g} \, 1 - \exp[-2\pi \sqrt{R_x/(\hbar \omega - E_g)}]}$$

(6.30)

For still higher photon energies, one reaches the continuum states. According to eqn (6.22e), the envelope function is expressed as

$$|F(0)|^2 = \frac{z \exp(z)}{\sinh z} = \frac{\pi \sqrt{R_x}}{\sqrt{\hbar \omega - E_g} \, 1 - \exp[-2\pi \sqrt{R_x/(\hbar \omega - E_g)}]}$$

where $z = \pi \gamma = \pi/a_B k$.

The absorption coefficient may be calculated in the usual way. In the calculation the following integral is involved:

$$I = \int \delta \left[ \frac{\hbar^2 k^2}{2\mu} - (\hbar \omega - E_g) \right] \frac{4\pi k^2 \,dk}{1 - \exp[-2\pi/a_B k]}.$$
The integral may be evaluated easily by using the properties of the $\delta$-function. The final result is

$$
\alpha_{\text{con}}(\hbar \omega) = A \frac{2\pi \sqrt{R_x}}{1 - \exp[-2\pi \sqrt{R_x/((\hbar \omega - E_g))]}, \quad (6.31)}
$$

where $A$ is a constant that precedes the $(\hbar \omega - E_g)^{1/2}$ term in the expression for the band-to-band absorption coefficient.

$$
\alpha_{\text{free}}(\hbar \omega) = A(\hbar \omega - E_g)^{1/2}.
$$

A look at eqn (6.31) reveals that close to the absorption fringe $\hbar \omega - E_g \ll R_x$ and $\omega \gg 1$ and the effect of the Coulomb interaction is quite strong. In the absorption continuum also, the values of $\alpha_{\text{con}}$ are significantly larger than $\alpha_{\text{free}}$. To see this effect, let $\hbar \omega - E_g = \pi^2 R_x$; the corresponding $\omega$ is 1 and absorption coefficient is still larger by a factor of 4.6. When $(\hbar \omega - E_g) = 20R_x$ the enhancement is 1.5. Comparing eqns (6.31) and (6.32) the absorption coefficient in the continuum may be written as

$$
\alpha_{\text{con}} = \alpha_{\text{free}} C(\omega) \quad (6.33a)
$$

where

$$
C(\omega) = \frac{2\pi \sqrt{R_x}}{1 - \exp[-2\pi \sqrt{R_x/((\hbar \omega - E_g))}] \cdot \frac{1}{\sqrt{\hbar \omega - E_g}}} \quad (6.33b)
$$
Ideal excitonic absorption spectra for transitions to discrete levels (vertical lines) and to continuum states (continuous curve).

6.5 Experimental absorption spectra for GaAs for different temperatures (reproduced with permission; after M. D. Sturge 1962, Phys. Rev. 127, 768).
Optische Eigenschaften von Halbleitern

Polaritonen: Hybridzustände aus Materie und Licht

\[ \alpha_{k}^{\dagger} \beta_{-k'}^{\dagger} : \]

\[ B_{\nu,K}^{\dagger} = \sum_{k,k'} \delta[K-(k-k')] \psi_{\nu} \left( \frac{k+k'}{2} \right) \alpha_{k}^{\dagger} \beta_{-k'}^{\dagger} \]

\[ = \sum_{k} \psi_{\nu} (k-K/2) \alpha_{k}^{\dagger} \beta_{k-k}^{\dagger} . \]
The derivation of this relation is best obtained in the Dirac representation, where

$$ B_{\nu, \mathbf{K}}^\dagger = |\nu \mathbf{K}\rangle \langle 0| $$

$$ \sum_{k, k'} |k, -k'\rangle \langle k, -k'| = 1 $$

$$ B_{\nu, \mathbf{K}}^\dagger = \sum_{k, k'} |k, -k'\rangle \langle k, -k'| |\nu \mathbf{K}\rangle \langle 0| $$

$$ = \sum_{k, k'} \langle k, -k'| |\nu \mathbf{K}\rangle |k, -k'\rangle \langle 0| $$

$$ = \sum_{k, k'} \langle k, -k'| |\nu \mathbf{K}\rangle \alpha_k^\dagger \beta_{-k'}^\dagger $$

$$ \langle k, -k'| |\nu \mathbf{K}\rangle = \int d^3r d^3r' \langle k, -k'| \mathbf{r}, \mathbf{r}' \rangle \langle \mathbf{r}, \mathbf{r}'| |\nu \mathbf{K}\rangle $$

$$ = \int d^3r d^3r' e^{-ik \cdot r} e^{i k' \cdot r'} e^{i \mathbf{K} \cdot (\mathbf{r} + \mathbf{r}')/2} \psi_\nu(\mathbf{r} - \mathbf{r}') $$

$$ = \delta[\mathbf{K} - (k - k')]|\psi_\nu\left(\frac{k + k'}{2}\right) $$

$$ \psi_0(k) = 8\sqrt{\pi a_0^3} \frac{1}{[1 + \left(ka_0\right)^2]^2} \quad \text{in 3D} \]
The Hamiltonian for free excitons can be written as

$$\mathcal{H}_0 = \sum_k \hbar e_{\nu k} B_{\nu,k}^\dagger B_{\nu,k}.$$

$$\mathcal{H}_I = -\sum_{k,q} d_{cu} \left[ \alpha_{\frac{1}{2} q + k}^\dagger \beta_{\frac{1}{2} q - k}^\dagger \mathcal{E}(q)e^{-i\omega_q t} + \text{h.c.} \right].$$

$$\sum_{\nu} \psi^*_\nu (\kappa) B_{\nu,K}^\dagger = \sum_{\nu k} \psi^*_\nu (\kappa) \psi_{\nu} (k - K/2) a_{k}^\dagger \beta_{K - k}^\dagger$$

$$= \sum_k \delta_{k,k - K/2} a_{k}^\dagger \beta_{k - k}^\dagger = a_{\frac{1}{2}K + \kappa}^\dagger \beta_{\frac{1}{2}K - \kappa}^\dagger.$$

$$\mathcal{H}_I = -\sum_{k,q,\nu} d_{cu} \left[ \psi_{\nu} (k) B_{\nu,q}^\dagger \mathcal{E}(q)e^{-i\omega_q t} + \text{h.c.} \right].$$

$$\mathcal{E}(q,t) = \mathcal{E}(q)e^{-i\omega_q t} = i\sqrt{2\pi\hbar\omega_q}(b_q - \text{h.c.}) \quad \omega_q = \frac{cq}{\sqrt{\varepsilon_0}}.$$

$$\mathcal{H}_I = -i\hbar \sum_{\nu q} g_{\nu q} (B_{\nu q}^\dagger b_q - \text{h.c.}) \quad \hbar g_{\nu q} = d_{cu} \psi^*_\nu (r = 0) \sqrt{\pi\hbar\omega_q / 2}.$$
\[
\mathcal{H} = \hbar \sum_q \left[ \sum_{\nu} e_{\nu q} B_{\nu q}^\dagger B_{\nu q} + \omega_q b_{q}^\dagger b_{q} - i \sum_{\nu} g_{\nu q} (B_{\nu q}^\dagger b_{q} - \text{h.c.}) \right].
\]

\[p_q = u_q B_q + v_q b_q, \quad [p_q, p_q^\dagger] = |u_q|^2 + |v_q|^2 = 1.\]

\[\mathcal{H} = \hbar \sum_q \Omega_q p_q^\dagger p_q.\]

Evaluating the commutator \([p, \mathcal{H}]/\hbar\): \[
\Omega_q p_q = \Omega_q (u_q B_q + v_q b_q)
= u_q (e_q B_q - i g_q b_q) + v_q (\omega_q b_q + i g_q B_q).
\]

\[0 = (\Omega_q - e_q) u_q + i g_q v_q\]

\[0 = -i g_q u_q + (\Omega_q - \omega_q) v_q,\]

\[(\Omega_q - e_q)(\Omega_q - \omega_q) - g_q^2 = 0.
\]

Polariton spectrum

\[\Omega_{q,1,2} = \frac{1}{2} (e_q + \omega_q) \pm \frac{1}{2} \sqrt{(e_q - \omega_q)^2 + 4g_q^2}.
\]
\( u_{q,1} = \sqrt{\frac{\Omega_{q,1} - \omega_q}{2\Omega_{q,1} - e_q - \omega_q}} \) and \( v_{q,1} = i\sqrt{\frac{\Omega_{q,1} - e_q}{2\Omega_{q,1} - e_q - \omega_q}} \)

\( u_{q,2} = \sqrt{\frac{\Omega_{q,2} - \omega_q}{2\Omega_{q,2} - e_q - \omega_q}} \) and \( v_{q,2} = -i\sqrt{\frac{\Omega_{q,2} - e_q}{2\Omega_{q,2} - e_q - \omega_q}} \).
Polaritons: interacting waves of excitons and light
Fig. 13.15. The polariton dispersion in CdSe with an equidistant ruling of the $k$-axis (a) and the observed (b) and calculated structure (c) of the resulting Fabry–Perot modes. According to [75K2]
Fig. 13.18. The dispersion relation of the lowest free exciton resonance in CuCl (a) and the group velocity determined from the time-of-flight of picosecond laser pulses and from the dispersion relation (b) [79M1]
Fig. 5.5. The schematic explanation for the occurrence of propagation quantum beats (a) and their observation in Na vapor (b) in the ortho exciton resonance of the yellow series in Cu$_2$O (c) and in an ensemble of $^{57}$Fe nuclei (d) ([84R1,91F1,99B1])